Induction

Lecture 2 Jan 17, 2021

Proof by Induction

- **SET UP**: Consider a statement/formula that involves an integer N, and suppose we can prove the followings:

(1) The statement is correct for some initial value N_0

(2) Whenever the statement is true for N, it will also be true for N+1

Then (1) and (2) imply that the statement is TRUE for all $N \geq N_0$.

What is going on? First we know the statement is true for N_0 by (1). Then applying (2), we know it is true for $N_0 + 1$. Applying (2) again, we know it is true for $(N_0+1) + 1 = N_0 + 2$. Applying (2) repeatedly we find that the statement is true for all $N \geq N_0$

Proof by Induction

- **Equivalent Setup**: Consider a statement/formula that involves an integer N, and suppose we can prove the followings:

(1) The statement is correct for some initial value N_0

(2) Whenever the statement is true for all $1, 2, \ldots, N$, it will also be true for N+1

Then (1) and (2) imply that the statement is TRUE for all $N \geq N_0$.

Examples

Q1. For all positive integers N, show that $1+2+3+\cdots+N=\frac{N(N+1)}{2}$ ★

Answer: We prove ★ by Induction.

(1) We check that the statement is true for $N_0=1$:

Putting N=1 in the left-hand side of ★ we get 1

Putting N=1 in the right-hand side of ★ we get $\frac{1(1+1)}{2}$ = 1

So ★ is correct when N=1

(2) We <u>show</u> if ★ is true for N, it will be remain true if we change N to N+1

Step (2)

$$1 + 2 + \cdots + N + (N + 1) = (1 + 2 + \cdots + N) + (\mathrm{N} + 1) =$$

$$\frac{N(N+1)}{2} + (N + 1) = \frac{N(N+1) + 2(N+1)}{2} =$$

This is where the use the assumption
that the statement is true for N

$$\frac{(N+2)(N+1)}{2} \checkmark$$

Similar examples to practice at home

Q. Show that

- $1^2 + 2^2 + 3^2 + \ \ldots + N^2 = \frac{N(N+1)(2N+1)}{6}$
- $1 + 4 + 7 + \ \ldots + (3N - 2) = \frac{N(3N-1)}{2}$

Examples

- **Q2.** Consider the famous Fibonacci sequence $\{x\_n\}_{n=1,2,3\ldots.}$, defined by the relations

- $x_1 = 1, x_2 = 1,$

- and $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$

Show that

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad ★$$

Answer. Before starting the proof, lets calculate some of x_n for small n

$$x_3 = x_1 + x_2 = 2, \qquad x_4 = 3, \qquad x_5 = 5, \qquad x_6 = 8, \qquad x_7 = 13, \qquad \ldots$$

It is a magic that the formula ★ always yields integers!

Examples

- **Answer.** We prove ★ by induction on n.

- (1) Putting n=1 in the formula we get $x_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = \frac{1}{\sqrt{5}}\frac{2\sqrt{5}}{2} = 1$ ✓

- (2) Assuming the formula is correct for n, we show it is also correct for

We have

$$x_{n+1} = x_n + x_{n-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] =$$

$$\frac{1}{\sqrt{5}}\left[\left(1+\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(1+\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$

Examples

Check that

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = 1 + \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = 1 + \frac{1-\sqrt{5}}{2}$$

Which gives us

$$\frac{1}{\sqrt{5}}\left[\left(1+\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(1+\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Examples

- **Q3.** Prove that $a^2 - 1$ is divisible by 8 for all odd integers a

Solution. First, since $a^2 = (-a)^\wedge 2$, we may assume a is a positive integer

Every positive odd integer has the form $a = 2n - 1$ for some $n \geq 1$

We prove the statement by induction on n.

1) Putting $n = 1$, we get $a = 1$ and $a^2 - 1 = 0$ which is divisible by 8 ✓

2) If the statement is true for a=2n-1, we show it will be also true for a+2= 2(n+1)-1:

$$(a+2)^2 - 1 = a^2 + 4a + 4 - 1 = a^2 - 1 + 4(a+1) = a^2 - 1 + 8n$$

Since $a^2 - 1$ and $8n$ are both divisible by 8, their sum will also be divisible by 8

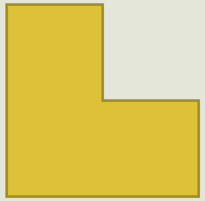
Similar examples to work at home*

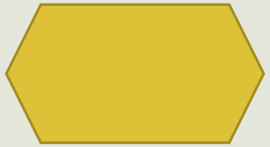
- Prove that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for all n ≥ 0.
- Prove that $a^4 - 1$ is divisible by 16 for all odd integers a.
- Prove that $a^{2n} - 1$ is divisible by 4×2^n for all odd integers a, and for all integers n.
- Prove that $n^3 + 2n$ is divisible by 3 for all integers n.
- Prove that $17n^3 + 103n$ is divisible by 6 for all integers n.
- Prove that $2^n + 1$ is divisible by 3 for all odd integers n

* Questions are taken from https://www.math.waikato.ac.nz/~hawthorn/MATH102/InductionProblems.pdf

Examples in geometry

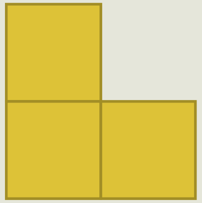
- **Q4.** Prove that a convex n-gon can be divided into n−2 triangles.

A 6-gon which is not convex 

A 6-gon which is convex 

Examples in geometry

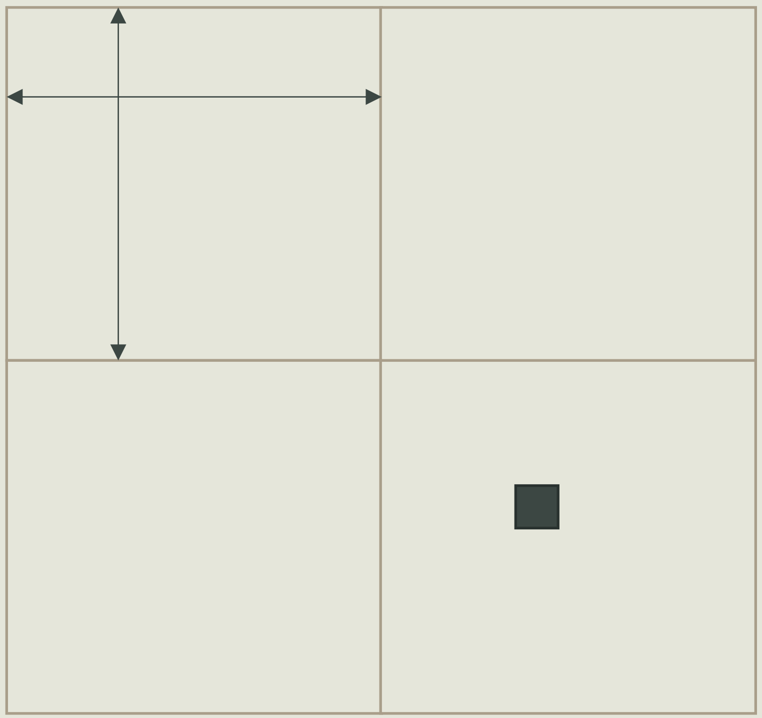
- **Q4.** Prove that a $2^n \times 2^n$ board with a single square missing can be covered using L-shaped pieces:



- **Answer.** We prove this by induction

- For n=1 : a 2x2 board with one square removed is precisely the L-shaped piece

- Assuming the statement is true for n and show that it is true for n+1



More questions to think at home

- **Q6.** For what natural numbers $n \geq 1$, it is possible divide a square into n squares (of not necessarily equal sizes).
- **Q7.** Prove, using induction, that $\sqrt{2}$ is not a rational number